Small Disturbances in Turbomachine Annuli with Swirl

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A general theory is proposed for the small disturbance field in strongly swirling flows in turbomachines. In contrast to the situation in nonswirling flows, vorticity, pressure, and entropy fields are not independent. Shear disturbances are not purely convected but rather propagate slowly in flows stable in the sense of Rayleigh, and are unstable in flows approaching free vortices. In the course of the propagation, there is an interchange between radial and tangential velocity perturbations. General conditions for propagation of pressure modes in the strongly swirling flow show the effects of rotation to be small if the frequency is large compared to base rotational frequency. Oscillatory shear flows have been observed in experiments in the MIT Blowdown Compressor.

I. Introduction

OUR theoretical and intuitive understanding of flow over wings and bodies depends to a considerable extent on the separation of a general small disturbance field in an otherwise uniform flow into three distinct classes of disturbance, namely entropy, vorticity, and pressure disturbances, which do not interact to first order except at flow discontinuities. The pressure disturbances propagate as sound, while the entropy and vorticity are purely convected, as shown by Kovasznay.[1] It is this separation into noninteracting modes which allows us to ignore the viscous and thermal wakes of a wing in computing its pressure field, for example. The superposition of pressure disturbances to represent arbitrary potential flows forms the basis of linear aerodynamic theory. Vorticity perturbations represent shear flows and turbulence, which are free of pressure fluctuations to first order, and the entropy mode represents temperature fluctuations other than those associated with the pressure mode.

These simplifications due to uncoupling do not necessarily apply in flows with strong rotation such as are typical and in fact essential in turbomachines, the reason in simplest terms being that density perturbations and radial or tangential velocity perturbations generate radial and tangential accelerations through unbalanced centrifugal and Coriolis forces. Thus it is readily seen that a viscous wake shed from a compressor rotor blade cannot be treated as a shear region of constant static pressure, in contrast to the wake of an aircraft wing, because its radial force imbalance inevitably results in a pressure disturbance coupled to the shear disturbance.

The idea motivating this work is that a better comprehension of turbomachine flows should follow from a clearer understanding of the behavior of the small disturbances in the rotating flowfield of a turbomachine. Some new phenomena, not present (or controlling) in external flows should appear as results of the strong rotation. Even a casual perusal of the extensive literature of rotating flows, starting with Proudman[2] and Taylor[3] will suggest many possibilities not accounted for in most current models of turbomachine flows. Much recent work in rotating flows however has been focused on confined flows and geophysical flows,[4] and is not directly applicable to turbomachines, where swirl velocities are of the same order as throughflow velocities.

The present discussion will be restricted to flow regions far enough from the surfaces of the machine that viscous effects can be neglected. Thus we will be concerned with the behavior of a disturbance field in the strongly swirling flow between blade rows, but will not at this time attempt to deal with the mechanisms which produce the disturbance field. As an example, we will describe the behavior of a blade wake, but not the growth of the boundary layer from which it originates.

The importance of rotational effects can be assessed tentatively in terms of the two dimensionless groups which are uniquely important in rotating flows, namely the Ekman number, $E=\nu/\Omega L^2$, and the Rossby number, $\epsilon = V/\Omega L$, where Ω is the angular velocity of flow, L is the length scale, V is the velocity scale of the disturbance, and ν is the kinematic viscosity. In this treatment, we shall argue that E is of the order of the inverse Reynolds number for the disturbance, hence small in regions remote from the disturbance's origin, so that we may neglect the viscous effects peculiar to rotation. The Rossby number, on the other hand, measures the importance of the inertial effects of rotation. Given Ω, the relative importance of such effects is seen to depend on the magnitude of V/L for the different types of disturbances, where V/L may be interpreted as a frequency or angular velocity. For a given length scale, L, the importance of rotational effects increases as the ratio of the disturbance frequency to the rotational frequency decreases. As the sound field of a turbomachine has its largest energy content at frequencies several times the shaft frequency, and many times the mean swirl frequency, we might expect rotational effects on the sound field to be minimal, and indeed this will be demonstrated below. For "nearly convected" disturbances, in contrast, the frequency is zero (in the natural coordinate system) in the absence of rotational effects, so rotation should play an important role in the behavior of viscous wakes for example. In the limit of strong rotation and slow flow, one has the axially uniform flow which we characterize as Taylor-Proudman flow. The behavior of wakes in a turbomachine lies somewhere between this limit and the purely convected limit of external flows, exhibiting the phenomenon of shear waves, which does not exist for nonrotating flows. The character of these waves can be seen by considering the behavior of a velocity disturbance in a base flow with solid-

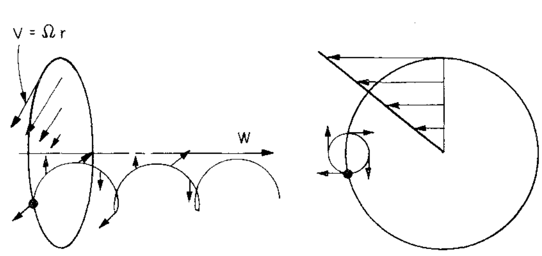

Fig. 1 Schematic of a turbomachine annulus (compressor) indicating the base flows of interest, a solid body rotation in region 1, and a mixed solid body-free vortex in region 2.

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body rotation, as sketched in Fig. 1. If a typical fluid element, indicated by the black dot, is given a small velocity outward relative to the mean flow, then as it proceeds outward conserving angular momentum it develops a tangential velocity defect relative to the mean flow, which results in a lag in its rotation hence an inward radial acceleration due to the unbalance of pressure and centrifugal forces. Thus the particle tends to describe the circular path indicated, relative to the mean flow. Its relative rotation is opposite the rotation of the mean flow. This characteristic motion is independent of whether the initial velocity perturbation is radial or tangential; furthermore it is independent of the detailed form of the base rotational flow, so long as the circulation increases outward. The shear, or inertial waves which result from the Coriolis and centrifugal forces have been extensively discussed in the literature of rotating fluids.[4]

Most analyses of small disturbances in turbomachine annuli can be grouped into two classes, those which are basically acoustic, and those describing an axisymmetric throughflow. In the former group, which includes the work of McCune,[5] Okurounmu and McCune,[6] and Lordi,[7] the flowfield of a blade row is represented by a superposition of isentropic pressure disturbances which are in fact the acoustic modes of the annulus. In the work of Tyler and Sofrin,[8] the focus was on the propagation characteristics of these same acoustic modes. Such treatments neglected the effects of strong rotation. We shall find that this neglect is justified only for disturbances with frequencies many times the fluid rotational frequency. The theory of axisymmetric throughflow as presented by Marble[9] considers the effects of strong rotation which give rise to the classical streamline shifts found in actuator disk theory. We shall see, however, that the assumption of no azimuthal variations eliminates many important effects of the rotation. In particular, oscillations arise in the axisymmetric flow only for swirl angles much larger than are common in turbomachines, whereas they occur for all swirl angles when angular variations are included.

Recently, McCune and Hawthorne,[10] and Cheng[11] have developed linearized theories in which the expansion is about a strong free vortex flow. The treatment is inviscid, and since it deals with the flow through the blade row as well as downstream of it, it is limited to Beltrami flows in which the disturbance vorticity is aligned with the flow and the flow is irrotational. The strong boundary layer-wake flows which appear to dominate three dimensional effects in turbomachines are thus excluded.

Stability plays an important role in rotating flows, and is the subject of an extensive literature. Usually, the question of stability is posed in terms of the temporal growth or decay of a small disturbance in a rotating fluid, without a large axial velocity. In the context of turbomachinery, where the major disturbances originate in blade rows through which the fluid streams, stability may be connected to the growth or decay of these disturbances as they are transported downstream by the axial flow. Chandrasekhar[12] has noted the equivalance of these two points of view, and his analogy will be exploited in the discussions of stability which follow. In some cases, instability appears to manifest itself in unsteady behavior of the flow through the blade row.

II. General Formulation

The geometry sketched in Fig. 2 will be assumed. The flow is bounded radially by a cylindrical casing and hub, and passes the stationary guide vanes (or nozzles in a turbine), then the rotor, and finally the stator. In region 1, the tangential velocity is approximated by $V=\Omega r$ in a compressor, and may also be rotational in a turbine, while usually in region 2, $V\approx\Omega r+\Gamma/r$. We will assume that the mean flow variables are independent of z and θ in all of the regions, but variable in r in accordance with radial equilibrium. There may be radial entropy gradients, especially in turbines. To avoid the complexities of standing wave phenomena, the treatment of disturbances in any one region will assume that the region is semi-infinite in z. Furthermore, we shall not address the complete boundary-value problem even in this limited context; rather, we study the types of disturbances which can exist in the annulus subject to only the conditions at r_0 and r_i, and attempt to catalog a set of disturbances which is complete in the sense that by linear superposition it can represent an arbitrary set of boundary conditions at the blade (r,θ,o) plane. The quantitative matching of these disturbances to exit conditions at the blade row will be left for other efforts.

The steady (unperturbed) flow has axial velocity $W(r)$, tangential velocity $V(r)$, and entropy $S(r)$. Denoting the total value of each of the variables by a prime, we expand about the mean values, so that the velocity components are $u'=u, v'=V+v, \quad w'=W+w$. Similarly the entropy, pressure, density, and temperature are $s'=S+s, \quad p'=P+p$, $\rho'=R+\rho$, $t'=T+t$. A consistent mean flow need satisfy only the relations

$$\frac{1}{R}\frac{dP}{dr}=\frac{V^2}{r} \tag{1}$$

$$S=c_p(\log T-\frac{\gamma-1}{\gamma}\log P) \tag{2}$$

$$P=\frac{\gamma-1}{\gamma}c_pR\,T \tag{3}$$

If the flow has constant stagnation temperature, there is the added constraint $T_t=T+(W^2+V^2)/2c_p$, which connects $T(r)$ to $W(r)$ and $V(r)$.

The mean quantities being functions only of r, we may expect solutions harmonic in z, θ, τ, which denotes time. Thus, carrying out a Fourier analysis in these variables, we represent the dependent variables by

$$q(r,\theta,z,z)=\int q(r,m,k,\omega)e^{i(kz+m\theta-\omega\tau)}\,dk\,dm\,d\omega \tag{4}$$

The following system of ordinary differential equations for ρ, u, v, w, s, then results

$$i\lambda u-\frac{2V}{r}v=-\frac{a^2}{R}\frac{d\rho}{dr}+[(2-\gamma)\frac{V^2}{r}-\frac{a^2S'}{c_p}]\frac{\rho}{R}-\frac{a^2}{c_p}\frac{ds}{dr}-\frac{\gamma V^2}{c_pr}s \tag{5}$$

$$i\lambda v+(V'+\frac{V}{r})u=-i\frac{ma^2}{r}(\frac{s}{c_p}+\frac{\rho}{R}) \tag{6}$$

$$i\lambda w+W'u=-ika^2(\frac{s}{c_p}+\frac{\rho}{R}) \tag{7}$$

$$i\lambda\rho+R\frac{du}{dr}+(R'+\frac{R}{r})u+\frac{imR}{r}v+ikRw=0 \tag{8}$$

$$i\lambda s+S'u=0 \tag{9}$$

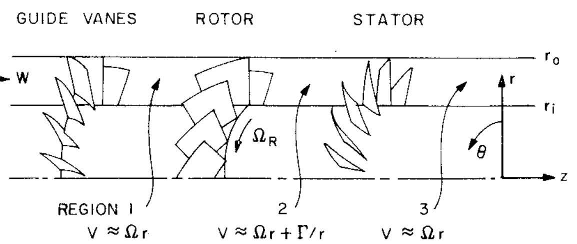

Fig. 2 Illustrating the mechanisms for oscillation in a solid body base flow (top) and in a negative entropy gradient (bottom).

where $\lambda = kW + mV/r - \omega$ is the eigenvalue of the convective operator $L \equiv \partial/\partial\tau + (V/r)\partial/\partial\theta + (W)\partial/\partial z$.

These five equations can be reduced to a second-order equation in u, plus a system of algebraic and first-order differential relations coupling v, w, ρ, and s to u, as follows. First eliminate s with Eq. (9) and solve Eqs. (6) and (7) for v and w. Then solve Eq. (8) for ρ and finally, eliminate all but u from Eq. (5) to find after some manipulation

$$\{\frac{r}{\lambda^2 A}\exp\int \frac{\gamma M_s^2}{r} dr u'\}' + \frac{r}{\lambda^2 A}\exp\int \frac{\gamma M_s^2}{r} dr H(r) u = 0 \tag{10}$$

where $A = 1 - m^2a^2/\lambda^2 r^2 - k^2a^2/\lambda^2$.

$$H(r) \equiv \mu' + A[\lambda^2/a^2 - 2M_s^2/r^2 - 2VV'/a^2 r]$$
$$+ \mu[2mV/\lambda r^2 + (\gamma - 1)M_s^2/r - (\lambda A)'/\lambda A]$$
$$+ A(M_s^2/r)(S'/c_p)$$
$$\mu \equiv \frac{1}{r} + M_s^2/r - kW'/\lambda - m(V' + V/r)\lambda r$$
$$M_s \equiv V/a.$$

The boundary conditions will be taken of the form,

$$u(r_i) + \alpha_i u'(r_i) = 0 \tag{11a}$$

$$u(r_o) + \alpha_o u'(r_o) = 0 \tag{11b}$$

and in many cases $\alpha_i = \alpha_o = 0$ wil be adequate.

Equations (10) and (11) constitute a Sturmian system provided the coefficient

$$r \exp\int \frac{\gamma M_s^2}{r} dr/\lambda^2 A$$

is real and does not vanish, hence may be considered positive in the interval $r_i < r < r_o$. This requires that λ be real, a condition which is satisfied for the oscillatory classes of disturbances, but not necessarily when k is complex.

The general character of the eigenvalue problem can be outlined as follows. $H(r)$, $\lambda(r)$ are known functions of ω and m, which may be regarded as determined by the source of excitation, so that only the axial wave number, k, remains undetermined and is to be considered the eigenvalue. Eigenfunctions satisfying Eq. (10) must oscillate in r, with more oscillations in $r_i < r < r_o$ as the radial order of the eigenfunction increases. The representation of an arbitrary disturbance field requires at least one infinite set of such eigenfunctions, corresponding to an infinite set of k. For real k, the solutions are oscillatory in z, otherwise they are exponential in z or unstable, as will be shown for special cases. Although general oscillation theorems exist,[13] which limit the number of zeros of the solution given the behavior of the coefficients in Eq. (10), they are difficult to apply to this very general equation because of the complexity of $H(r)$.

A direct numerical approach to computation of the eigenfunctions of the annulus with arbitrary swirl, axial velocity, and entropy gradients, could be based on Eqs. (10) and (11). For the present purposes, further simplifications seem in order.

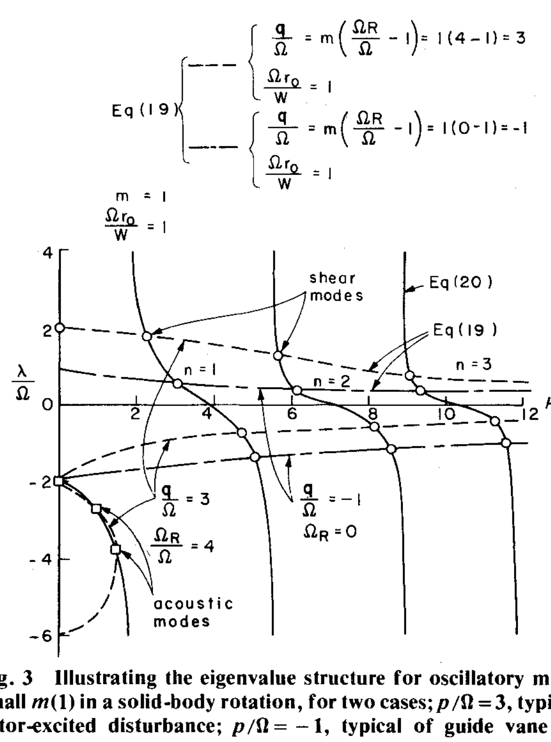

Fig. 3 Illustrating the eigenvalue structure for oscillatory modes of small $m(1)$ in a solid-body rotation, for two cases; $p/\Omega = 3$, typical of a rotor-excited disturbance; $p/\Omega = -1$, typical of guide vane disturbances.

Isentropic Flow

Uniform gas ingested by an ideal (lossless) compressor would satisfy the conditions $S' = 0$, and in the limit of low Mach number, $s = 0$, so that the special case of a completely isentropic flow is of some general interest. Although it should be possible to derive a general equation for u from Eq. (10) for the isentropic case, it appears easier to obtain an expression in ρ, since for $S' = s = 0$, Eqs. (5-7) are algebraic in u, v, w, and can be solved to yield

$$u = \frac{i}{D}\left\{\frac{\lambda a^2}{R}\frac{d\rho}{dr} + [\frac{2mVa^2}{Rr} - (2-\gamma)\frac{\lambda V^2}{Rr}]\rho\right\} \tag{12}$$

$$v = -\frac{1}{D}\left\{(V' + \frac{V}{r})\frac{a^2}{R}\frac{d\rho}{dr} - [(2-\gamma)\frac{V^2}{Rr}(V' + \frac{V}{r}) - \frac{\lambda m a^2}{Rr}]\rho\right\} \tag{13}$$

$$w = (\frac{i}{\lambda}W')u - (\frac{k}{\lambda}\frac{a^2}{R})\rho \tag{14}$$

where $D = \lambda^2 - (2V/r)[V' + (V/r)] \neq 0$.

It is readily demonstrated that this velocity field is irrotational if V is irrotational, i.e., if $V = \Gamma/r$, but it is otherwise rotational. Since the flow is barotopic, the total fluid acceleration has a potential, but this does not mean that the velocity perturbations are irrotational. Indeed this is only the case when the mean flow is irrotational.

Eliminating u, v, w, from Eq. (8), and expanding the coefficients to second order in the swirl Mach number gives the following equation for ρ,

$$\{\frac{r}{D}\exp\int \frac{(2\gamma - 3)M_s^2}{r} d r \rho'\}' + \frac{r}{D}\exp\int \frac{(2\gamma - 3)M_s^2}{r} dr K(r)\rho = 0 \tag{15}$$

where

$$K(r) \equiv [\frac{D}{a^2}(1 - \frac{a^2k^2}{\lambda^2}) - \frac{m^2}{r^2} - \frac{2mV}{\lambda r^3}(1 + \frac{rD'}{D} - \frac{rV'}{V}) - \frac{2kmV}{\lambda^2 r^2}W']$$

Its implications will be explored for Regions 1 and 2.

III. Region 1—Wheel Flow

This case, which is algebraically the simplest, represents adequately the flow after guide vanes or perhaps turbine nozzles. We put

$$V=\Omega r, \qquad \lambda=kW+m\Omega-\omega$$

$$D=\lambda^2-4\Omega^2, \quad a^2=a_c^2+\frac{\gamma-1}{2}\Omega^2r^2$$

where a_c is the sound speed at $r=0$. It will also be assumed that W is constant, even though this implies a radial variation in stagnation temperature, which is not entirely realistic for flows induced by inlet guide vanes. The elimination of W' however makes it possible to see the effects of rotation in the simplest context.

With these conditions, Eq. (15) becomes

$$r^2\frac{d^2\rho}{dr^2}+[1+(2\gamma-3)\frac{\Omega^2r^2}{a^2}]r\frac{d\rho}{dr}+\left[(\frac{\lambda^2-4\Omega^2}{a_c^2})\right.$$

$$\left.(1-\frac{k^2a_c^2}{\lambda^2})r^2-\frac{\gamma-1}{2}\frac{\Omega^2r^4}{a_c^2}-m^2\right]\rho=0 \qquad (16)$$

From Eq. (12) for the radial velocity, the boundary conditions for hard walls at r_o,r_i are

$$\frac{r}{\rho}\frac{d\rho}{dr}=-\frac{2m\Omega}{\lambda}+\frac{\Omega^2r^2}{2a_c^2}, \quad r=r_o,r_i \qquad (17)$$

A consistent solution of this system of equations which is correct to second order in the swirl Mach number requires inclusion of the terms in Ω^2r^2/a_c^2 in the coefficients of $d\rho$ / dr and ρ respectively. No solution in terms of tabulated functions has occurred to the author, and at this stage, numerical development of solutions has not seemed warranted, since it does not seem that elucidation of the principal effects of swirl should require inclusion of terms of second order in the swirl Mach number. Accordingly, we drop the terms of order Ω^2r^2/a_c^2, and the solution of Eq. (16) is simply

$$\frac{\rho}{R}=Z_m(\mu r)=AJ_m(\mu r)+B\,N_m(\mu r) \qquad (18)$$

where

$$\mu^2=(\frac{\Omega}{W})^2[4-(\lambda/\Omega)^2][1-M^2+2(\frac{q}{\Omega})(\frac{\Omega}{\lambda})$$

$$+(\frac{q}{\Omega})^2(\frac{\Omega}{\lambda})^2] \qquad (19)$$

and $q=\omega-m\Omega$, $M=W/a_c$.

The boundary conditions are

$$\frac{\mu r}{mZ_m}Z_m'(\mu r)=-2\frac{\Omega}{\lambda}, \quad \mu r=\lambda r_o, \quad \mu r_i \qquad (20)$$

Here we regard Ω, m, and q as set by the physical description of the problem, since for example we might have $q=m(\Omega_R-\Omega)$ where Ω_R is the angular velocity of a rotor if we deal with region 1. Therefore Eqs. (19) and (20) represent an eigenvalue problem for λ/Ω.

For the sake of simplicity in the illustration, we take the special case of $r_i/r_o=0$, when $Z_m=AJ_m$ since the N_m are infinite at $\mu r=0$. The eigenvalue problem is most clearly seen by rewriting Eq. (20) as

$$\frac{\lambda}{\Omega}=\frac{2Z_m}{\frac{\mu r}{m}Z_{m+1}-Z_m}$$

and plotting this relation together with Eq. (19) as functions of μr. The first relation depends only on m, and is shown as the solid line in Figs. 3 and 4, which are for $m=1$ and $m=10$ respectively. This relation gives $\lambda/\Omega=-2$ for $\mu r=0$ for all m, goes to $-\infty$ for μr on the order of m, then has successive zeros corresponding to the zeros of Z_m. The second relation, (19) gives μr_o as a function of $(\Omega r_o/W)^2$, M and q/Ω as well as of λ/Ω, so that to display it we must choose values for $(\Omega r_o/W)$, q/Ω and M. Since $q/\Omega=m(\Omega_R/\Omega-1)$, it represents the frequency of the disturbance measured in a coordinate system rotating with the fluid. For a rotor-produced disturbance in the guide vane-rotor combination of a high-tip speed compressor, we might have $q/\Omega\approx 2m$. On the other hand for the disturbance produced by the guide vanes alone, $\Omega_R=0$, (i.e., $\omega=0$), and $q/\Omega=-m$. The parameter $(\Omega r_o/W)^2$ measures the magnitude of the swirl velocity relative to axial velocity. Typically for compressors it is less than unity, but can be as large as 2.5 for the region between turbine nozzles and rotors. We will use a value of unity for purposes of illustration. Similarly an axial Mach number $M=0.5$ is selected.

The eigenvalue problem is illustrated for $m=1$ in Fig. 3 and for $m=10$ in Fig. 4, for the two cases of $q/\Omega=3$ and $q/\Omega=-1$. Referring first to Fig. 3, we see that for $q/\Omega=3$, there are two distinct sets of solutions, those for $|\lambda/\Omega|<2$, which are marked by the circles, and those for $\lambda/\Omega<-2$, marked by the squares. The former, to be termed shear waves, comprise an infinite set with both positive and negative values of λ/Ω. They represent the swirling-flow analog of vorticity as they have low frequencies, of the order of Ω. The latter are the swirling flow equivalent of acoustic waves, and will be termed pressure waves. For the example of Fig. 3, only two such propagating modes exist, all other pressure disturbances being attenuated in z.

Referring next to Fig. 4, it can be seen that the character of the shear waves has not changed greatly except that the first pair now occurs for $\mu r_o\approx 14$, whereas for $m=1$ they occurred for $\mu r_o\approx 4$. For large m, the first pair of roots occurs for $\mu r_o\approx m$. Two propagating pressure wave pairs now exist, instead of one as in Fig. 3.

For the guide-vane generated disturbances, which correspond to $q/\Omega=-m$, there are only shear waves, but a doubly infinite set of these for both $m=1$ and $m=10$. It is convenient to discuss the various disturbance types according to the range of $|\lambda/\Omega|$ in which they fall.

A. Shear Waves $[0<|\lambda/\Omega|<2]$

As noted, these disturbances form a doubly infinite set, one positive and one negative. From the definition of λ,

$$kr_o=\frac{\Omega r_o}{W}(\frac{q}{\Omega}+\frac{\lambda}{\Omega})$$

and recalling that $\lambda/\Omega=0$ corresponds to a convected disturbance, we see that the positive and negative values of λ/Ω give two waves propagating oppositely in the fluid. Since $|\lambda/\Omega|<2$, while $q/\Omega=\omega/\Omega-m$ is of order m, the deviation from pure convection decreases as m as one would expect, since for $m\to\infty$ a two-dimensional limit is approached.

The general characteristic equation for λ_{mn} follows from putting $u=0$ for $r=r_o$, r_i. It can be written as

$$Q(\mu r_o)/Q(\mu r_i)=1 \qquad (21)$$

where

$$Q(\mu r)=\frac{(\mu_{mn}r)J_m'+(2m\Omega/\lambda_{mn})J_m}{(\mu_{mn}r)N_m'+(2m\Omega/\lambda_{mn})N_m}$$

In the limit of large m, the $(\mu r_o)_{mn}$ are nearly the roots of $J_m(\mu r_o)_{mn}=0$, and simple forms can be obtained for λ_{mn} and the other quantities of interest thus,

$$\left.\begin{aligned}\frac{\lambda_{mn}}{2\Omega}&\simeq\pm\frac{qr_o}{(\mu r_o)_{mn}W}\\ kr_o&\simeq\frac{qr_o}{W}[1\pm\frac{2\Omega r_o}{(\mu r_o)_{mn}W}]\end{aligned}\right\} \quad m\gg 1 \qquad (22)$$

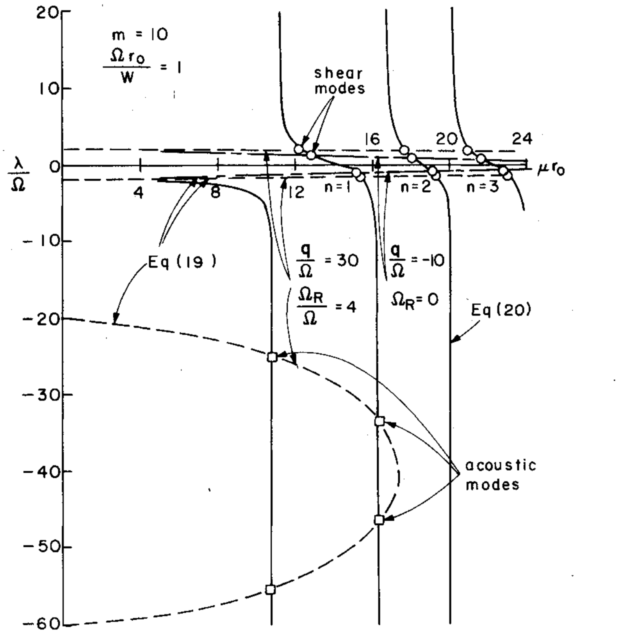

Fig. 4 Illustrating the eigenvalue structure for oscillatory modes of large $m(10)$, in a solid body rotation, for two cases; $p/\Omega = 30$, typical of a rotor-excited disturbance; $p/\Omega = -1$, typical of guide vane disturbances.

and the velocity components are

$$u \simeq \frac{-ia^2(\mu r_o)_{mn}}{2\Omega r_o[1-(\lambda_{mn}/2\Omega)^2]}\left\{\frac{m}{(\mu r_o)_{mn}}Z_m + \frac{\lambda_{mn}}{2\Omega}Z'_m\right\} \quad \text{(23a)}$$

$$v \simeq \frac{a^2(\mu r_o)_{mn}}{2\Omega r_o[1-(\lambda_{mn}/2\Omega)^2]}\left\{\frac{\lambda_{mn}}{2\Omega}\frac{m}{(\mu r_o)_{mn}}Z_m + Z'_m\right\} \quad \text{(23b)}$$

$$w \simeq \frac{a^2}{W}\left[1 + \frac{q}{2\Omega}\left(\frac{2\Omega}{\lambda_{mn}}\right)\right]Z_m \quad \text{(23c)}$$

Some features of the velocity and pressure field of the shear waves are readily seen from Eq. (23). Recalling Eq. (18), we see that ρ, v, w are all in phase, while u is shifted in phase by $\pi/2$ in $m\theta$ or kz as indicated by the i. Since $\lambda_{mn}/2\Omega \rightarrow 0$ for the higher radial modes $u \propto Z_m$ for these modes, while $v \propto Z'_m$; $w \propto 2\Omega/\lambda_{mn}$, hence large. The magnitude of the pressure (or density) field is best seen by noting that $\rho R \sim Z_m$, while $v/\Omega r_o \sim [a^2/(\Omega r_o)^2](\mu r_o)_{mn} Z_m$ so that $\rho/R \sim (\Omega r_o/a)^2 (v/\Omega r_o)/(\mu r_o)_{mn}$. That is, the fractional density perturbation is of the order of the swirl Mach number squared times the velocity, but reduced by the factor $1/(\mu r_o)_{mn}$, which varies as $1/m$ for large m. For typical values of $\Omega r_o/a \approx 1/2$, $m \approx 30$, $\rho/R \approx 0.008(v/\Omega r_o)$. If $v/\Omega r_o \approx 0.1$, this implies an acoustical level of 134dB at atmospheric pressure. This is a modest level for the interior of a turbomachine, so it appears the principal importance of the shear waves in region 1 lies in the rotor's interaction with their velocity field, not in their pressure field.

To illustrate the functional behavior of the velocity components and density for smaller m, an example corresponding to the first radial mode ($n=1$) of Fig. 4 is shown in Fig. 5. All velocities and the density are shown to the same (arbitrary) scale. We see that for this mode the disturbances are small except in the outer half of the duct, implying that the form would be similar for r_i/r_o as large as 0.4.

B. Shear Waves [$\lambda/\Omega = \pm 2$]

From Eq. (19) there is a solution corresponding to $\lambda/\Omega = \pm 2$, for which $\mu r = 0$, and the solution becomes a limiting form of the Bessel function. Although ρ satisfies a

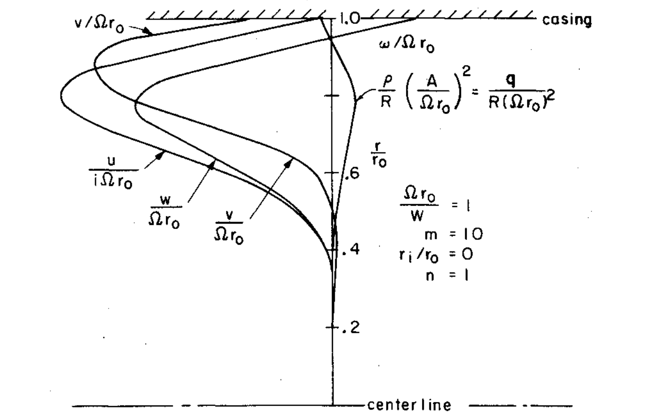

Fig. 5 An example of the velocity and pressure structure of shear waves in a solid body rotation, corresponding to the first radial mode ($n=1$) of Fig. 4.

second-order differential, u is governed by a first-order equation, hence u cannot satisfy the two point boundary-value problem, and $\lambda/\Omega = \pm 2$ does not yield a solution.

C. Purely Convected Disturbances [$\lambda = 0$]

It seems there is only one class of such shear disturbances, for which $\lambda = 0$, namely those for which $u = v = \rho = 0$, but $w \neq 0$. Continuity then requires that $k = 0$, so these are disturbances with axial velocity perturbations only, arbitrary in r and θ, but uniform in z. Such shear regions might be formed by a nonlifting blade rotating with (i.e., at the same angular velocity as) a solid body rotation.

D. Pressure Waves $\left[\frac{-q/\Omega}{1-M} < \frac{\lambda}{\Omega} < \frac{-q/\Omega}{1+M}\right]$

The range of λ/Ω for these disturbances lies between the lower two zeros of μr_o as a function of λ/Ω in Figs. 3 and 4. As noted above, these disturbances are the swirl-flow analog of acoustic waves, and are in fact only slightly modified by swirl in most situation because $|q/\Omega|$ and hence $|\lambda/\Omega|$ is large. The set of propagating modes is finite, since the curve of μr_o has a maximum, at $\lambda/\Omega = -(q/\Omega)/(1-M^2)$, given by

$$(\mu r_o)_{max} = r_o q/a(1-M^2)^{1/2}$$

A mode which falls below this value in μr_o will propagate, otherwise it will be attenuated in z. This is the generalization to swirling flows of the familiar cutoff condition of acoustic theory. Solving Eq. (19) for λ_{mn} for $(\lambda/\Omega)^2 \gg 1$ gives

$$\frac{\lambda_{mn}}{2m\Omega} = \frac{-q \pm \sqrt{q^2M^2 - (1-M^2)\mu_{mn}{}^2W^2}}{2m\Omega(1-M^2)} \quad \text{(24)}$$

which together with Eq. (21) determines the eigenvalues. In the limit of q/Ω large, $\lambda_{mn}/\Omega \rightarrow \infty$, and Eq. (21) reverts to

$$J'_m(\mu r_i)J'_m(\mu r_o) - N'_m(\mu r_i)N'_m(\mu r_o) = 0$$

which is the usual result for nonswirling flows.

In the rotating fluid, the pressure mode will be above cutoff, i.e., propagating, if λ_{mn} is real, and this condition becomes simply

$$(\omega - m\Omega)^2M^2 < (1-M^2)\mu^2W^2$$

Now if we identify $\omega = m\Omega_R$, the condition for propagation can be written

$$\left[\frac{(\Omega_R - \Omega)r_o}{a_c}\right]^2 > (1-M^2)\left(\frac{\mu r_o}{m}\right)^2 \quad \text{(25)}$$

The left side is the square of the tangential relative Mach number. For m large, $\mu r_o/m$ is of order unity, for the first radial mode (see Fig. 4) and then increases with increasing n if we denote the radial eigenvalue by n. For the first mode, the cutoff condition is roughly that the *total* relative Mach number be less than unity.

The argument is readily extended to include complex conjugate values of $(-\lambda/\Omega)$ by noting that for $(-\lambda/\Omega)$ large, μ_{mn} is nearly independent of $(-\lambda/\Omega)$, and equal to the root of $Z_m=0$. If $(-\lambda/\Omega)$ is complex, then from Eq. (20) $\mu r Z_m'(\mu r)/Z_m$ must also be complex, but the imaginary part is small, so we can estimate the imaginary part of λ_{mn} by assuming that μ_{mn} in Eq. (24) has the value corresponding to $Z_m=0$. Then λ_{mn} is given by

$$\lambda_{mn}=\frac{\omega-m\Omega}{2m\Omega(1-M^2)}\pm i\frac{\sqrt{(1-M^2)\mu_{mn}{}^2W^2-(\omega-m\Omega)^2M^2}}{2m\Omega(1-M^2)} \tag{26}$$

These modes attenuate exponentially in z, since $kW=\lambda-m\Omega+\omega$ is now complex. With the propagating modes found for $\mu r<(\mu r)_{\max}$ they form the analog of the pressure mode for nonswirling flows.

E. An Example

To illustrate the behavior of the two major classes, shear waves and pressure waves, they are indicated in a cascade representation of a stage in Fig. 6. The example chosen is $m=1$, $\Omega r_o/w=1$, corresponding to the eigenvalue structure of Fig. 3. In region 1, the two shear waves shed by the inlet guide vanes are indicated at the left, their lines of constant phase being shown by the dashed lines to either side of the convective direction. In this region there is also the upstream propagating pressure wave from the rotor, indicated by the solid lines of constant phase at the right.

If we think of an $m=1$ viscous disturbance shed from the guide vanes (due for example to an inlet distortion pattern) as being made up of a sum of the upstream and downstream propagating shear waves, it becomes clear that the disturbance will be dispersed by the differential propagation of its components. Higher harmonies would have still different rates of upstream and downstream propagation, the speed relative to the fluid decreasing with increasing m.

If the rotor is nonlifting, so that $V=\Omega r$ also in Region 2 then the above analysis also applies there. A pressure wave propagates downstream, and there is a shear wave pair, shown again by the dashed lines of constant phase to either side of the convective direction.

Although this discussion has been for isentropic flow, it is readily seen from Eq. (9) that an entropy disturbance introduced into a mean flow with $S'=0$, by shear heating or heat transfer on the blade, will be purely convected, so should be found to preserve its structure, in contrast to the viscous disturbance.

A more detailed study of shear waves in a solid body rotation has been carried out by Yousefian,[14] including solution of the problem of matching to the wakes of an inlet guide vane row. He has also shown that under conditions of very large swirl ($\Omega r_o/w \gg 1$), upstream propagating wakes can form, after the fashion of "Taylor Columns."

IV. Region 2: $V=\Omega r+\Gamma/r$

A typical base flow in Region 2 may be modeled as a sum of the wheel flow, which persists from Region 1, and a vortex flow, such as would be generated by a rotor producing a radially constant stagnation temperature rise. Thus we take $V=\Omega r+\Gamma/r$. The special case of $\Gamma=0$ has been discussed above, so the main emphasis here will be on the effects of the free vortex. Consistent with the simplification introduced in Region 1, we take W constant. Then in Eq. (15), $\lambda=kW+m\Omega-\omega+m\Gamma/r^2$, and $D=\lambda^2-4\Omega^2 4\Omega\Gamma/r^2$. Since λ and D are both functions of r, the governing equation is more complicated than that for the solid-body base flow. No solutions in terms of tabulated functions are available for the general case.

Reasoning by analogy to the case of wheel flow, we may expect solutions of at least two types, one being essentially acoustic, slightly modified by the effects of rotation, and the second being nearly convected, analogous to the shear waves. If this analogy applies, we would expect the distinction between modes to lie in the magnitude of $\lambda/[\Omega+\Gamma/r^2]$, this being the ratio of the disturbance frequency as measured in coordinates moving with the fluid, to the local base flow rotational frequency. If this ratio is large, essentially acoustic behavior is expected. If it is of order unity, we expect the disturbances to show strong effects of rotation. But since

$$\frac{\lambda}{\Omega+\Gamma/r^2}=m+\frac{kW-\omega}{\Omega+\Gamma/r^2}$$

where m and $kW-\omega$ are constants, it is readily seen that the frequency ratio can be of order unity over the whole annulus only if a) m is of order unity, b) $\Gamma/r^2 \ll \Omega$, or c) the range of r is small, i.e., $r_o/r_i \sim 1$. Otherwise any given mode will show low frequency behavior at the radii where the frequency ratio is of order unity, and high frequency behavior elsewhere. The larger m, the narrower is the range of r in which low frequency behavior will exist, since for large m, $\lambda/(\Omega+\Gamma/r^2)$ is a difference of two large quantities if it is small. We consider first the behavior of high-frequency disturbances.

A. Pressure Waves $V=\Gamma/r$

The effects of a solid body rotation having been examined above, it will be sufficient to consider the case of $\Omega=0$ for the high frequency class of disturbances. From Eq. (15), taking $\lambda=kW+m\Gamma/r^2-\omega$, $D=\lambda^2$, the basic equation governing the density is

$$r^2\rho''+[1+2\gamma-3)\frac{\Gamma^2}{r^2}+\frac{4m\Gamma}{\lambda r^2}]r\rho'+[\frac{r^2\lambda^2}{a^2}$$
$$-r^2k^2-m^2-\frac{4m\Gamma}{\lambda r^2}+\frac{8m^2\Gamma^2}{\lambda^2 r^2}]\rho=0 \tag{27}$$

For high frequency disturbances, as explained above, $|m\Gamma/\lambda r^2| \ll 1$, and we may expand it as

$$\frac{m\Gamma}{\lambda r^2}=\frac{m\Gamma}{(kW-\omega)^2r^2+m\Gamma}\approx\frac{m\Gamma}{(kW-\omega)^2r^2}$$
$$[1-\frac{m\Gamma}{(kW-\omega)^2r^2}+\ldots]$$

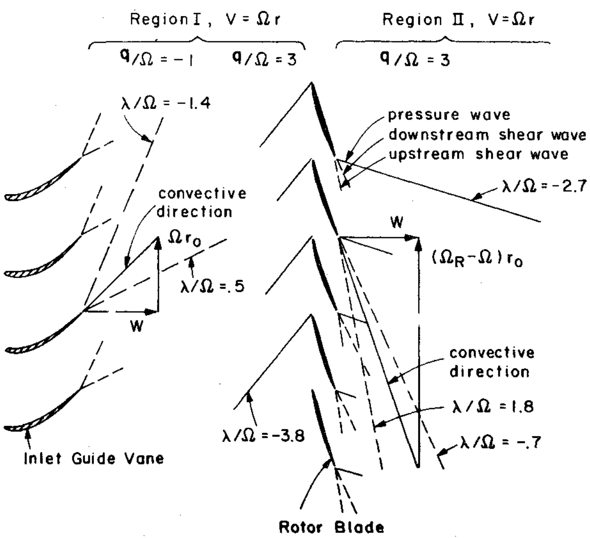

Fig. 6 Illustrating the disturbances in regions 1 and 2 (for $\Gamma=0$) corresponding to the eigenvalue structure of Fig. 3.

and neglect the second term. For the same reasons, we neglect the last term in the coefficient of ρ. Finally, the term $(2\gamma-3)\Gamma^2/r^2$ being of the order of the square of the swirl Mach number, it is also neglected. The solution is then

$$\rho = e^{-m\Gamma/(\omega-kW)r^2} Z_\nu(\alpha r) \tag{28}$$

where

$$\nu^2 = m^2 + 2(\omega-kW)m\Gamma/a^2 \text{ and } \alpha^2 = (\omega-kW)^2/a^2 - k^2$$

From Eq. (12), if we require $u=0$, $r=r_o$, r_i the condition determining the radial eigenvalues to first order in Γ is simply $Z_\nu'(\alpha r)=0$, $r=r_o$, r_i, which is precisely the condition for the case of zero swirl. Therefore, the only effect on the eigenvalue comes from the change in the order, ν, of the $Z_\nu(\alpha r)$.

To estimate the effect of this change in boundary conditions on α and hence on the axial wave number we represent the roots of $Z_\nu'(\alpha r)=0$ by an expansion around their values for zero swirl, putting

$$(\alpha r_o)_{\nu n} = (\alpha r_o)_{mn} + \beta_{mn}(\nu - m) + \ldots$$

where

$$\beta_{mn} = \partial(\alpha r_o)_{mn}/\partial m$$

From the relation for ν, $\nu - m \simeq (\omega-kW)\Gamma/a^2$, and we find for the axial wave number

$$(kr_o)_{mn}(1-M^2) = -M\left(\frac{\omega r_o}{a} - \frac{\beta\Gamma}{ar_o}\right)$$

$$\pm\sqrt{\left(\frac{\omega r_o}{a}\right)^2 - \frac{2(\alpha r_o)(\omega\beta\Gamma)}{a^2} - (1-M^2)(\alpha r_o)^2} \tag{29}$$

The condition for propagation is that the square root be real, or

$$\left(\frac{\omega r_o}{a}\right)^2 \geq (1-M^2)(\alpha r_o)^2 + \frac{2(\alpha r_o)(\omega\Gamma\beta)}{a^2} \tag{30}$$

where the ω in the last term may be taken as that for $\Gamma=0$ in this approximation.

For large m, $\beta \simeq 1$, and $\alpha r_o/m \simeq 1$, so that this condition becomes approximately $(\omega r_o/ma)^2 > 1 - M^2 + 2\omega\Gamma/ma^2$, and if we identify $\omega = m\Omega_R$, we see that to first order in $\Gamma/r_o a$ this is the condition that the rotor tip relative Mach number exceed unity. Thus, to first order in the swirl Mach number, the condition for pressure wave propagation is the same in the free vortex as in the solid body flow.

When $(kr_o)_{mn}$ according to Eq. (29) is complex, the pressure mode is exponentially attenuated in z, i.e., the mode is "below cutoff." This will be true for sufficiently large αr_o for any given M and $(\omega r_o/a)$, so that all radial modes with αr_o above some values are below cutoff, as for the solid body rotation.

B. Shear Disturbances $V = \Omega r + \Gamma/r$

To treat these disturbances, on which the rotational effects should be strong, we return to Eq. (15), and let $M_s^2 \to 0$, thus suppressing compressibility effects. It is convenient to define a new coordinate, $\xi^2 = \Omega r^2/\Gamma$. With some manipulation Eq. (15) can then be written,

$$\xi^2 p'' + \left[1 + 4\frac{m^2 + (m\delta - 2)\xi^2}{D\xi^4/\Omega^2}\right]\xi p' + \left\{-m^2 - \left(k^2\frac{\Gamma}{\Omega}\right)\xi^2 \times \frac{D\xi^4/\Omega^2}{(\delta\xi^2+m)^2} + \frac{4m^2[1+(2-\delta/m)\xi^2]}{D\xi^4/\Omega^2}\right\}p = 0 \tag{31}$$

where $\delta = kW/\Omega + m - \omega/\Omega$ is the non-r dependent part of λ/Ω, i..e, $\lambda/\Omega = \delta + m/\xi^2$ and

$$D\xi^4/\Omega^2 = -[(4-\delta^2)\xi^4 + 2(2-m\delta)\xi^2 - m^2] \tag{32}$$

From Eqs. (12-14) the velocities are

$$\frac{uR\sqrt{\Omega\Gamma}}{i} = \frac{\Omega^2}{D}\left\{\frac{\lambda}{\Omega}p' + \frac{2m}{\xi}\left(1+\frac{1}{\xi^2}\right)p\right\} \tag{33a}$$

$$vR\sqrt{\Omega\Gamma} = -\frac{\Omega^2}{D}\left\{2p' + m\left(\frac{\lambda}{\Omega}\right)p\right\} \tag{33b}$$

$$wR\sqrt{\Omega\Gamma} = -\frac{k}{\lambda}\sqrt{\Omega\Gamma}p \tag{33c}$$

In addition to the singular point at $\xi=0$, the coefficients of Eq. (31) are singular at three points. $\xi_c^2 = -m/\delta$ is the point at which a disturbance is purely convected, since $\lambda/\Omega = 0$ there. The other two singularities are at the roots of $D=0$. Since from Eq. (33) both u and v are singular at these points, they must be excluded (at least provisionally) from the solution

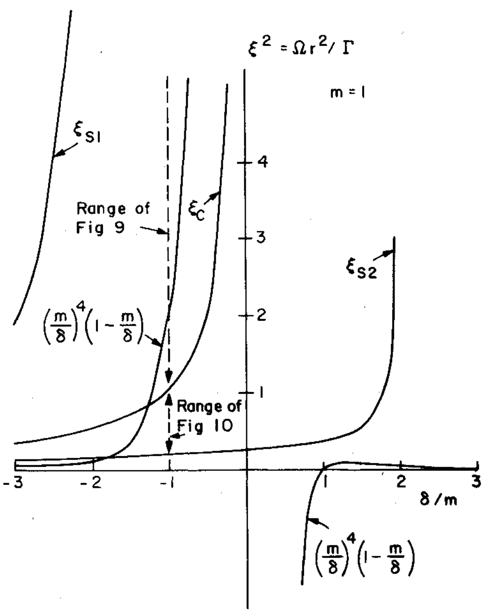

Fig. 7 Loci of the singular points of Eq. (31), for $m=1$ showing the ranges of ξ^4 and δ/m in which oscillatory "shear" disturbances exist, and the location of the examples in Figs. 9-12.

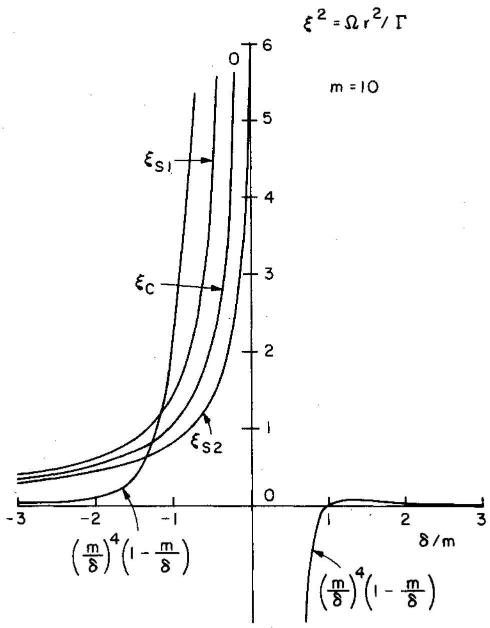

Fig. 8 Loci of the singular points of Eq. (31) for $m=10$ showing the ranges of ξ^2 and δ/m in which oscillatory "shear" disturbances exist to be much narrower than for $m=1$ (Fig. 7).

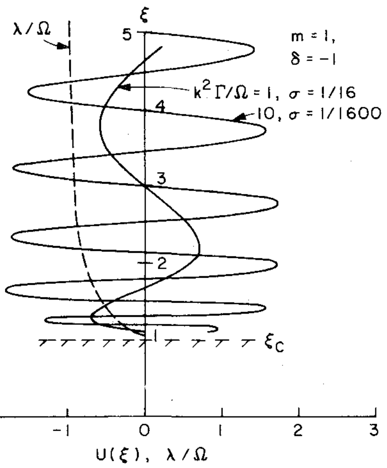

Fig. 9 Showing the radial velocity structure of a shear mode having $m=1, \delta=-1$, and $\xi_c^2<\xi^2<\xi_{sI}^2$ (Fig. 7), for two values of $k^2\Gamma/\Omega$.

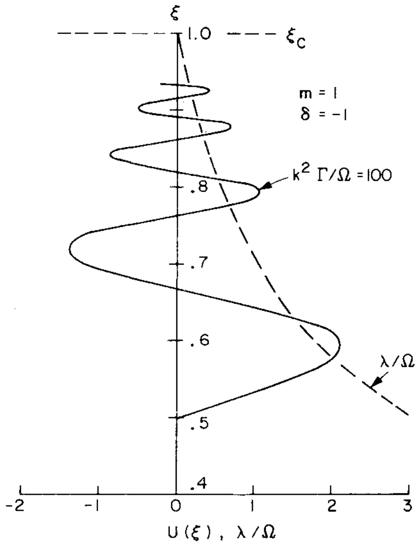

Fig. 10 Showing the radial velocity structure of a shear mode having $m=1, \delta=-1$, and $\xi_{s2}^2<\xi^2<\xi_c^2$ (Fig. 7).

domain. These points, denoted ξ_{sI} and ξ_{s2}, span ξ_c, as shown in Fig. 7 for $m=1$ and in Fig. 8 for $m=10$.

Only for $\delta<0$ will the eigenfunction approach closely to convective behavior; since for $\delta>0$, ξ_c is imaginary.

The range of ξ^2 is set by characteristics of the base flow, since

$$\xi^2=\Omega r^2/\Gamma=(\Omega r_o/W)(r_oW/\Gamma)(r/r_o)^2$$

The inlet guide vanes set $\Omega r_o/W$, and the rotor pressure rise sets r_oW/Γ. For incompressible flow $\Delta p_t/\rho=\Omega_R\Gamma$, and we have

$$\xi^2=\frac{\rho(\Omega_R r_o)^2}{\Delta p_t}\left(\frac{\Omega}{\Omega_R}\right)\left(\frac{r}{r_o}\right)^2$$

For a very lightly loaded rotor operating in a solid body flow, $\Delta p_t/\rho(\Omega_R r_o)^2<1$ and ξ^2 is large for all r/r_o. On the other hand a heavily loaded rotor operating in a weak prerotation field will have low values of ξ^2, approaching zero in the absence of guide vanes. From Figs. 7 and 8, we see that for $\xi_{s2}^2<\xi^2<\xi_{sI}^2$ δ/m can have a wide range for $m=1$, but must be negative and near zero for $m=10$. In the limit of $\xi_{sI}\rightarrow\infty$, the corresponding $\delta_I\rightarrow-2$, while as $\xi_{s2}\rightarrow\infty$, $\delta_2\rightarrow 2$, and we have the limit of a pure solid body base flow discussed in Sec. III. In that case, all shear modes have real k and hence are oscillatory.

Returning to the general case, where ξ^2 can range from 0 to ∞, we note that the condition for existence of an infinite set of eigenfunctions is that there be an arbitrarily large number of zeros of p within the annulus, since u is coupled to p by the first-order relation, Eq. (33). If we adopt the comparison equation $a(\xi)p''+b(\xi)p'+c(\xi)p=0$, where $a(\xi)=\xi^2$, the local condition for oscillation is that $b^2-4ac<0$. Thus the requirement for an arbitrarily large number of zeros in the annulus is that $c(\xi)$ be sufficiently positive. Now $D(\xi_c^2)=-4(1+\xi_c^2)\Omega^2/\xi^2<0$ so that $D(\xi)<0$ for $\xi_{s2}<\xi<\xi_{sI}$, and $D(\xi)>0$ elsewhere. Further, $\delta/m<2$ for $\xi_{s2}<\xi<\xi_{sI}$, so the last term of $c(\xi)<0$ and if $c(\xi)$ is to be positive, the second term must be large and positive. This can occur in two ways, 1) for k real, $\xi_{s2}^2<\xi^2<\xi_{sI}^2$ and $\delta\xi^2+m$ small, i.e., for ξ sufficiently close to ξ_c; and 2) for $k=ik_I$, $0<\xi^2<\xi_{s2}^2$, in which case $D(\xi)>0$.

Two important differences between these two classes of solution will be noted. First, when k is real, v and w are in phase in θ or z, while u is shifted $\pi/2$; when k is imaginary, w is shifted $\pi/2$ relative to v (Eq. 33). Secondly, when k is imaginary, the solutions are unstable, as will be elaborated later.

Consider first the case where k is real. The point $\xi^2=\xi_c^2-m/\delta$ is a regular singular point, so that a power series solution of the form

$$p=\sum_{n=0}^{\infty}a_n(\xi-\xi_c)^{n+s}$$

is appropriate. The indicial equation yields $s=1/2\pm\sqrt{1/4-k^2f(\xi_c)}$, where $f(\xi_c)=-(\Gamma/\Omega)\xi_c^2[D(\xi_c)\xi_c^4/\Omega^2]/\delta^2$, so s is real only for $k^2f(\xi_c)<1/4$, a condition which can be rewritten as

$$\left(\frac{m}{\delta}\right)^4\left(1-\frac{m}{\delta}\right)<\frac{1}{16}\left(\frac{\Omega r_o^2}{\Gamma}\right)\left(\frac{m}{kr_o}\right)^2\equiv\sigma \qquad (34)$$

The left side of this inequality is plotted on Figs. 7 and 8, it is quite small except in the range $-2<\delta/m<0$, but this is the range of solutions with real k, at least from $m\gg 1$. Thus tentatively we conclude that the solutions are *irregular at ξ_c for real k*, except possibly for small m, and therefore ξ_c must lie outside the annulus for solutions with real k. It follows that the range of ξ^2 covered by a single eigenfunction with real k must be either $\xi_c^2<\xi^2<\xi_{sI}^2$ or $\xi_{s2}^2<\xi^2<\xi_c^2$. From Figs. 7 and 8 it is clear that these ranges are very wide for m small, but narrow for m large, so that *oscillatory shear flows are more likely to exist with small m than with large m.*

To illustrate this behavior, $u(\xi)$, obtained by numerical integration of Eq. (31) is plotted in Fig. 9 for $m=1$, $\delta=-1$ for two real values of $k^2\Gamma/\Omega$. We see that the period of oscillation in ξ decreases as $k^2\Gamma/\Omega$ increases, as expected, and that the period also decreases for $\xi\rightarrow\xi_c$, where $u(\xi)$ is singular because $(m/\delta)^4(1-m/\delta)>\sigma$. Since the exponent of the leading term in the power series expansion is imaginary, the behavior near ξ_c is of the form, $u\sim\cos[\log(\xi-\xi_c)]$. Similar oscillatory solutions exist for $\xi_{s2}<\xi<\xi_c$, an example being shown in Fig. 10 for the same values of $m=1$ and $\delta=-1$ as in Fig. 9. Here the domain is limited above by ξ_c.

For $m=1$, there are also oscillatory solutions for $\xi>0$. Here the range of ξ is limited below by ξ_{s2}.

As m is increased, the range of ξ allowing oscillatory solutions narrows, but the features are otherwise similar to those for m small.

It appears that for real k, there is an infinite set of eigenfunctions obtainable by varying $k^2\Gamma/\Omega$ for any given δ, so long as ξ lies entirely in either $\xi_c<\xi<\xi_{sI}$ or $\xi_{s2}<\xi<\xi_c$, but these ranges are too small to span the annuli of most compressors except for very small m. The boundary conditions at the rotor impose further limits on the choice of $k^2\Gamma/\Omega$ and δ, which will be discussed below.

There is an infinite set of eigenfunctions having $k=\pm ik_I$ and $0<\xi^2<\xi_{S2}^2$. For very small ξ^2, they exist for $-\infty<\delta<\infty$, while for large ξ^2, for $\delta>2$. There is a possibility of nearly, but not exactly, convective behavior in either situation. But if δ is real, we now have $\omega=\omega_R+i\omega_I$, where $\omega_I=k_I W$, and the disturbance takes the form

$$P=p(r)e^{i[m\theta-\omega_R\tau]\pm k_I[z-W\tau]} \tag{35}$$

These eigenfunctions are therefore *temporally unstable*. If matched to an arbitrary distribution of pressure (or velocities) at the plane $z=0$ at some time, say $\tau=0$, the disturbances would subsequently either grow or decay exponentially in time. To the extent that it must match to these disturbances, we conclude then that the flow through the blade row is unstable, hence must be unsteady in blade coordinates.

C. Shear Disturbances $V=\Gamma/r$ ($\xi^2\to 0$)

In this limit, which is of some special interest, $D=\lambda^2$, so that the three singular points of Eq. (31) converge to one and Eq. (31) reduces to

$$r^2p''+[1+4m\Gamma/\lambda r^2]rp'$$
$$+\{-m^2-k^2r^2-4m\Gamma/\lambda r^2+8m^2\Gamma^2/\lambda^2r^4\}p=0$$

where now $\lambda r^2=r^2(kW-\omega)+m\Gamma$. Both the second and last coefficients are singular at r_c, where $\lambda r^2=0$. Putting $x=r/r_c$, the equation becomes

$$x^2p''+[1-\frac{4}{x^2-1}]xp'+[-m^2-(kr_c)^2x^2$$
$$+\frac{4}{x^2-1}+\frac{8}{(x^2-1)^2}]p=0 \tag{36}$$

and it is clear that $x=1$ ($r=r_c$) is a regular singular point. Expanding about this point, we find two regular solutions, of the forms

$$p_1=(x-1)\sum_{n=0}^{\infty}a_n(x-1)^n$$

$$p_2=(x-1)^2\sum_{n=0}^{\infty}b_n(x-1)^n$$

so evidently in this limiting case of irrotational flow, the convective radius can be included in the annulus.

Application of oscillation theorems[13] to Eq. (36) shows that there are no solutions for real k. On the other hand by the arguments advanced above there is clearly an infinite set for $k=ik_I$. These are temporally unstable, having the form of Eq. (35). We conclude therefore that shear (low frequency) disturbances in a free vortex are unstable.

One exception to this statement must be noted, since from Eqs. (5-7) we find that for $V=\Gamma/r$, purely convected solutions can exist, with $\rho=s=0$, provided that $v=0$ as well. These "u, w only" disturbances are the analog for the free vortex of the "w only" disturbance noted for the wheel flow.

D. Matching to the Blade Row

The richness of possible solutions for Region 2 makes a complete analysis of the matching a formidable task, which will not be addressed here. Some discussion of the constraints imposed on the values of δ and $k^2\Gamma/\Omega$ is essential however.

The parameters determining the mean flow are $\Omega r_o/W$, Γ/r_oW and Ω_R/Ω. Then m and kr_o specify the eigenfunction, and the connection between these and the parameters of Figs. 7 to 12 is as follows,

$$\xi_o^2=(\frac{\Omega r_o}{W})/(\Gamma/r_oW)$$

$$\xi=\xi_o(r/r_o)^2$$

$$k^2\Gamma/\Omega=(kr_o)^2(\frac{\Gamma}{r_oW})(\frac{W}{\Omega r_o})$$

$$\delta=(kr_o)(\frac{W}{\Omega r_o})+m-(\omega_R+i\omega_I)/\Omega$$

where $\omega_R=m\Omega_R$

Now for *real* k, $\omega_I=0$, and we have

$$kr_o=(\frac{\Omega r_o}{W})[\delta-m+m\Omega_R/\Omega]$$

so for given m, $(\Omega r_o/W)$ and Ω_R/Ω, the range of values of kr_o is limited by the range of δ, which is sizeable for small m, but very small for large m. As an example, take $\Omega r_o/W=1$, $\Gamma/r_oW=.5$, $\Omega_R/\Omega=3$, then $\xi_o^2=2$, $kr_o=\delta+2m$, and for the values $m=1$, $\delta=-1$ of Fig. 9, $kr_o=1$ and $k^2\Gamma/\Omega=0.5$. If $-2<\delta<2$ (Fig. 7), $0<k^2\Gamma/\Omega<4$, so that the range of choice of $k^2\Gamma/\Omega$ is limited for the oscillatory disturbances.

For *imaginary* k, we must choose $\omega_I/\Omega=k_Ir_o(W/\Omega r_o)$, so that $\delta=m(1-\Omega_R/\Omega)$ is entirely determined but k_I can be chosen arbitrarily. For the above example, we would have $\delta/m=-2$, and it appears from Figs. 7 and 8 that we should be able to construct arbitrarily many solutions by varying k_I, so long as the range of ξ is either $\xi^2>\xi_{sI}^2$ or $0<\xi^2<\xi_{s2}^2$.

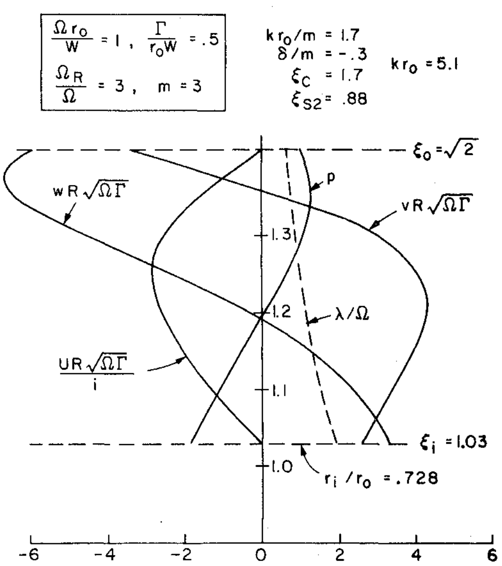

Fig. 11 Illustrating the velocity and pressure structures for a typical *oscillatory shear wave* in region 2, satisfying boundary conditions $u=0$ at $\xi_o=\sqrt{2}$, $\xi_2=1.03$ so that $r_i/r_o=.728$.

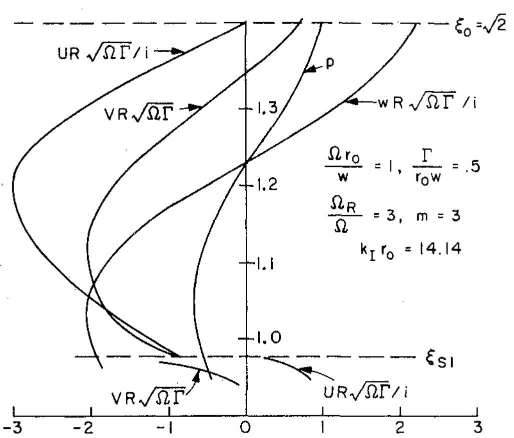

Fig. 12 Illustrating the velocity and pressure structures for a typical *unstable shear wave* in region 2, satisfying boundary conditions at $\xi_0=\sqrt{2}$ and (approximately) $\xi_i=1$, as in Fig. 11.

Solutions have been computed for the above example, with $m=3$ for both real and imaginary k. The solution for real k is shown in Fig. 11. Here $kr_o=5.1$ and the solution is a first radial mode for $\xi_i=1.03$ (or $r_i/r_o=.728$). Note that p, v, and w are all in phase (in θ or z) while u is shifted $\pi/2$ because it is imaginary.

A comparable solution for imaginary k is shown in Fig. 12. Here $k_I r_o=14.14$ was chosen, and this is a little small to give a zero above ξ_{sI}, so the solution shows a discontinuity in u at $\xi_{sI}=0.98$, but a slightly larger value of $k_I r_o$ would yield a solution satisfying the same boundary conditions as that of Fig. 11. Note that now both u and w are shifted $\pi/2$ in θ from p and v. There is an infinite set of such solutions with imaginary k, since $k_I r_o$ can be made arbitrarily large, independent of δ.

V. Conclusions

The primary conclusion which follows from this study is that small amplitude "shear" disturbances in strongly rotating base flows are not convected; rather they propagate slowly and may exhibit oscillatory behavior with a period of the order of the rotational period of the base flow. For any given fundamental mode there is an interchange between the radial and tangential velocity components during the oscillation. There is also a weak pressure field associated with these shear disturbances.

In a solid body rotation, all shear disturbances are oscillatory, while in a free vortex it appears that shear disturbances are generally unstable in the sense that any radial or tangential perturbation will lead to an unbounded displacement. More generally for base flows combining solid body and free vortex components, some shear disturbances will be oscillatory and some unstable. Oscillation is more likely for disturbances with small numbers of azimuthal periods.

When the shear disturbance is oscillatory, its radial velocity is shifted $\pi/2$ (in $m\theta+kz-\omega t$) from its tangential velocity, which is in phase with the axial velocity. In contrast, for convected disturbances all are in phase.

High frequency propagating or elliptic pressure fields such as represent the potential flowfield of a turbomachine rotor are not strongly influenced by base rotation. General conditions for acoustical cutoff in the strongly rotating flow are given; they reduce for large azimuthal periodicity to the condition that the tip relative Mach number be less than unity. This applies both upstream and downstream of a rotor.

There is a dearth of time resolved data with which to compare this theory; one set of data has recently been obtained in the MIT Blowdown Compressor, and a tentative comparison to the theory has been made.[15] In this compressor, the mean tangential velocity profile at outlet from the rotor has nearly a free-vortex form in the inner portion of the annulus, and nearly a solid body rotation in the outer portion. In accord with the theoretical predictions—a strong persistent shear disturbance is found in the outer portion, while in the inner portion the wakes dissipate very rapidly and entropy disturbances are convected. Furthermore the phase relationships between the pressure and the various velocity perturbations agree with the theory in the outer region.

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